Number of Subtractions in Fixed-Transfer Dispersion Relations*

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Assuming the minimal requirements necessary to derive the Froissart bound, the number of subtractions for the fixed-momentum-transfer dispersion relation in the unphysical region $0<t<4\mu^2$ turns out to be 2. In the proof, the positiveness of all the derivatives of absorptive part with respect to t at $t=0$ is used. Physical implications and applications of this result are briefly discussed.

IT has been shown by Froissart,[1] some time ago, that if a scattering amplitude satisfies Mandelstam representation, then the forward scattering amplitude (with relativistic normalization) is bounded by $Cs\ln^2(s/s_0)$ at high energies, where s is the square of the center-of-mass energy. Further, one of us[2,3] showed that the only necessary assumptions to get the Froissart result were:

(a) At fixed energy the scattering amplitude is analytic with respect to $t=-2k^2(1-\cos\theta)$, (where θ is the c.m. scattering angle and k the c.m. momentum) in some neighborhood $\mathfrak{D}$ of the segment $t=0\rightarrow t=4\mu^2$. Then it follows that the absorptive part of the amplitude is analytic inside an ellipse with foci $t=0$, $t=-4k^2$ and semimajor axis $t=2k^2+4\mu^2$. This may be shown because from the result of Lehmann[4] we know that the partial-wave expansion of the absorptive part converges in *some* ellipse, and due to the positiveness of the expansion coefficients, as a consequence of unitarity, the largest ellipse in which the expansion converges has a singularity at the extreme right in the t plane and therefore cannot intersect the segment $t=0$, $t=4\mu^2$. Further, it follows that the amplitude is analytic inside an ellipse with foci $t=0$, $t=-4k^2$ and semimajor axis

$$2k^2+\mu^2-\epsilon^a\,,\quad\text{where}\quad \epsilon\rightarrow 0 \text{ as } s\rightarrow\infty\,.$$

(b) The second assumption necessary for the proof is that for $0<t<4\mu^2$ the absorptive part of the amplitude is bounded by s^N. This latter assumption is familiar but it seems very hard to justify. [If one replaces this assumption by the much weaker condition $A(s,t)<\exp s^M$ for $t<1/s^N$ one still gets, adapting the argument of Refs. 2 and 3, that the forward scattering amplitude is polynomial bounded for real s.]

Here we want to maintain the minimal requirements necessary to derive the Froissart bound and take into account the further requirement that the scattering amplitude for fixed t, inside the region of analyticity in t described above ($\mathfrak{D}$), is analytic with respect to s in a twice-cut plane, with cuts from $s=(M_A+M_B)^2$ to $+\infty$ and from $s=-\infty-i\,\mathrm{Im}\,t$ to $s=(M_A-M_B)^2-t$, where M_A and M_B are the masses of the scattering particles.

Condition (b) enables us to write $N+1$ subtracted dispersion relations for the amplitude for $4\mu^2\geq t\geq 0$

$$F(s,t)=\sum_{n=0}^{N}C_n(t)s^n+\frac{s^{N+1}}{\pi}\int\frac{A_s(s',t)ds'}{s'^{N+1}(s'-s)}+\frac{u^{N+1}}{\pi}\int\frac{A_u(u',t)du'}{u'^{N+1}(u'-u)}\,,\quad(1)$$

where the familiar variable u is defined by

$$s+t+u=2(M_A^2+M_B^2)$$

and from unitarity $A_s(s,t)$, absorptive part associated with the reaction $A+B\rightarrow A+B$, and $A_u(u,t)$ associated with $A+\bar{B}\rightarrow A+\bar{B}$, are *positive*. This will turn out to be the crucial point of the present work.

For t inside $\mathfrak{D}$ and s complex, with say, $\mathrm{Re}s$ big enough, the integrals appearing in the right-hand side of (1) are uniformly convergent with respect to t and are therefore analytic functions of t, inside $\mathfrak{D}$ for fixed s. Therefore, the subtraction polynomial is itself analytic in t inside $\mathfrak{D}$, and since this is certainly true for $N+1$ values of s, this is also true for the coefficients $C_n(t)$.

However, following the lines of Ref. 2 or 3, it is easy to see that in addition to the information

$$|F(s,0)|<Cs\ln^2 s\,,\quad(2)$$

one can, given ϵ in advance, find a value $0<t_0<4\mu^2$ such that for $0\leq t\leq t_0$

$$|F(s,t)|<Cs^{1+\epsilon}.\quad(3)$$

We shall choose ϵ strictly less than unity.

Then in the interval $0\leq t\leq t_0$ we can write dispersion relations with two subtractions only:

$$F(s,t)=\alpha(t)+s\beta(t)+\frac{s^2}{\pi}\int\frac{A_s(s',t)ds'}{s'^2(s'-s)}+\frac{u^2}{\pi}\int\frac{A_u(u',t)du'}{u'^2(u'-u)}.\quad(4)$$

Expressions (1) and (4) should coincide for $0\leq t\leq t_0$.

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[1] M. Froissart, Phys. Rev. 123, 1053 (1961).
[2] A. Martin, Phys. Rev. 129, 1432 (1963).
[3] A. Martin, Lecture Notes at the Scottish Universities Summer School, 1963 (to be published).
[4] H. Lehmann, Nuovo Cimento 10, 579 (1958).

Using the familiar identity

$$\frac{X^2}{X'^2}\frac{1}{X'-X}=\frac{X^2}{X'^3}+\cdots+\frac{X^N}{X'^{N+1}}+\frac{X^{N+1}}{X'^{N+1}}\frac{1}{X'-X},$$

we get for $0\leq t\leq t_0$

$$\alpha(t)+s\beta(t)+\sum_{n=2}^{N}\left[\frac{s^n}{\pi}\int\frac{A_s(s',t)ds'}{s'^{n+1}}+\frac{u^n}{\pi}\int\frac{A_u(u',t)du'}{u'^{n+1}}\right]$$

$$\equiv\sum_{n=0}^{N}C_n(t)s^n. \quad (5)$$

Now let us distinguish two cases:

(i) N is even. Then clearly,

$$C_N(t)=\frac{1}{\pi}\int\frac{A_s(s',t)ds'}{s'^{N+1}}+\frac{1}{\pi}\int\frac{A_u(u',t)du'}{u'^{N+1}}. \quad (6)$$

This equation holds only for $t\leq t_0$, though $C_N(t)$ is analytic up to $t=4\mu^2$. However, from unitarity, we know that

$$A_s(s',t)=\sum_{n=0}^{\infty}A_s^{(n)}(s')t^n,$$

$$A_u(u',t)=\sum_{n=0}^{\infty}A_u^{(n)}(u')t^n,$$

with $A_s^{(n)}(s')$ and $A_u^{(n)}(s')\geq 0$.

Then, since the right-hand side of (6) converges for $t\leq t_0$, we conclude, according to a theorem on the exchange of summation and integration that for $t\leq t_0$

$$C_N(t)=\sum_{n=0}^{\infty}t^n\left[\frac{1}{\pi}\int\frac{A_s^{(n)}(s')ds'}{s'^{N+1}}+\frac{1}{\pi}\int\frac{A_u^{(n)}(s')ds'}{s'^{N+1}}\right].$$

However, this is the power-series expansion of $C_N(t)$, with positive coefficients. $C_N(t)$ being analytic up to $t=4\mu^2$, this expansion converges up to $|t|=4\mu^2$. Then reversing the argument, the convergence of the series guarantees the convergence of the two integrals in (6) from $t=0$ to $t=4\mu^2$. This means that we can undo one subtraction in integral representation (1).

(ii) N odd ≥ 3. Then from (5), using

$$u=2(M_A^2+M_B^2)-s-t$$

we get

$$C_N(t)=\frac{1}{\pi}\int\frac{A_s(s',t)ds'}{s'^{N+1}}-\frac{1}{\pi}\int\frac{A_u(u',t)du'}{u'^{N+1}}, \quad (7)$$

$$C_{N-1}(t)=\frac{1}{\pi}\int\frac{A_s(s',t)ds'}{s'^{N}}+\frac{1}{\pi}\int\frac{A_u(u',t)du'}{u'^{N}}$$

$$+N[2(M_A^2+M_B^2)-t]\frac{1}{\pi}\int\frac{A_u(u',t)du'}{u'^{N+1}}, \quad (8)$$

We notice that we have necessarily

$$2(M_A^2+M_B^2)\geq 4\mu^2,$$

otherwise we would have a singularity in t either at $t=4M_A^2$ or $t=4M_B^2$ due to the exchange of the $A\bar{A}$ or $B\bar{B}$ system and we would have to redefine μ. Then, noticing that the expansion coefficients of $[2(M_A^2+M_B^2)-t]^{-1}$ are all positive, we first prove that

$$C_{N-1}(t)[2(M_A^2+M_B^2)-t]^{-1}$$

has positive expansion coefficients in t. Then generalizing somewhat the argument of case (i), we deduce again that the three integrals in the right-hand side of (8) are convergent for $0\leq t<4\mu^2$. From this it follows immediately that

$$\int\frac{A_s(s',t)ds'}{s'^N}$$

also converges for $0\leq t>4\mu^2$. Hence, we can undo two subtractions.

Carrying again the process as many times as necessary, we conclude that representation (4), with only *two* subtractions holds not only for $t\leq t_0$ but on the whole segment $0\leq t<4\mu^2$, and the integrals

$$\int_{(M_A+M_B)^2}^{\infty}\frac{A_s(s',t)ds'}{s'^3},\quad \int_{(M_A+M_B)^2}^{\infty}\frac{A_u(u',t)du'}{u'^3}, \quad (9)$$

are absolutely convergent for $t<4\mu^2$.

If we make the further assumption that the subtraction coefficients $\alpha(t)$ and $\beta(t)$, which are already known to be analytic in $|t|<\mu^2$ are in fact analytic in $|t|<4\mu^2$, we can extend the validity of representation (4) to the whole region $|t|<4\mu^2$.[5] Indeed it is easy to see from unitarity that the expansions of $A_s(s',t)$ and $A_u(u',t)$ in power series around $t=0$ have positive coefficients, therefore, for $|t|<4\mu^2$ we have

$$|A_s(s',t)|\leq A_s(s',|t|),$$
$$|A_u(u',t)|\leq A_u(s',|t|).$$

And hence in region $|t|<4\mu^2$ only *two* subtractions are necessary.

Let us now list some of the consequences:

(a) From the assumptions made at the beginning of this paper the previous expression for the Froissart bound was, in the total cross section

$$\sigma_t(s)<C(N)\ln^2(s/s_0),$$

where $C(N)$ depends on the number of subtraction in the following way[2]:

$$C(N)=N^2(\pi/\mu^2)$$

However, from the convergence of integrals (9) we

[5] One can prove this, at least in some particular cases, by selecting an energy such that the size of the Lehmann ellipse is big enough.

now know that there is at least a sequence of increasing energies for which

$$\sigma_t(s) < (4\pi/\mu^2)\ln^2(s/s_0) \tag{10}$$

so that we have removed the major arbitrariness in the Froissart bound. In fact, a stronger condition can be obtained:

$$\int s^{-3}\exp\left[2\ln s\left(\frac{\mu^2}{4\pi}\frac{\sigma_t}{\ln^2 s}\right)^{1/2}-1\right]ds \text{ must } \textit{converge}. \tag{11}$$

This means that if on a segment s_1-s_2 σ_t exceeds (10) by a factor $1+\epsilon$ this segment must go to zero as s_1 goes to infinity.

(b) In the symmetric case where the scattering amplitude is invariant in the exchange of s and u (example $\pi^+\pi^0$ scattering) (4) can be rewritten in terms of the symmetrical variable $z=(s-2\mu^2+t/2)^2$

$$F(z,t)=f(t)+\frac{z}{\pi}\int_{(2\mu^2+t/2)^2}^{\infty}\frac{\mathrm{Im}F(z',t)dz'}{z'(z'-z)}. \tag{12}$$

From Eq. (12) we deduce the property $\mathrm{Im}F(z,t)/\mathrm{Im}z>0$ which is the definition of a "Herglotz" function.[6] Therefore, $F(z,t)$ has no complex zeros. We also notice that for z real $<[2\mu^2+(t/2)]^2$, $(d/dz)F(z,t)>0$ and thence $F(z,t)$ has at most one real zero (which corresponds to two complex conjugate zeros in the s plane). So the property $\mathrm{Im}F(s,t)>0$ for $4\mu^2\geq t\geq 0$ is also valid in the whole quadrant of the s plane $\mathrm{Im}s>0$, $\mathrm{Re}s>2\mu^2-t/2$ (with the corresponding symmetries). In addition, for $t\geq 0$ $F(s,t)$ cannot decrease faster than $1/s^2$ as s goes to infinity.[7]

(c) If analytic continuation to the third channel is possible, which is the case if Mandelstam representation is valid, one can investigate what happens for $t\rightarrow 4\mu^2$. Let us take the completely symmetric case in *stu*. Then the subtraction polynomial, taken at $t=4\mu^2$ gives us essentially the N first zero-energy scattering lengths in the t channel. These, if normal threshold behavior is assumed, are finite. Therefore the integrals

$$\int_{4M^2}^{\infty}\frac{A_s(s't)ds'}{s'^3} \text{ etc.}$$

have a finite limit for $t=4\mu^2$. They represent a particular case of the Froissart representation[8] of partial-wave amplitude in the t channel. Hence, from our result we deduce that the scattering lengths at $t=4\mu^2$ are analytic in the angular momentum l for $\mathrm{Re}l\geq 2$. More specifically if we define the scattering lengths as

$$a_l=\lim_{q\rightarrow 0}\frac{e^{i\delta_l(q)}\sin\delta_l(q)}{q^{2l+1}},$$

where q is the c.m. momentum in the t channel, and δ_l the phase shift in the t channel, the representation

$$a_l=\frac{1}{2M}\frac{\Gamma(l+1)}{\Gamma(l+3/2)\sqrt{\pi}}\int_{4M^2}^{\infty}\frac{A_s(s',4\mu^2)ds'}{s'^{l+1}}$$

is valid for $l=2$, 4, etc. $\cdots$, and since $A_s(s',4\mu^2)$ is positive we conclude that the scattering lengths are positive for $l\geqslant 2$. This could be extended to more realistic cases with sufficient care.

(d) More generally for $|t|<4\mu^2$ the holomorphy domain of the partial wave in the t channel certainly contains $\mathrm{Re}l\geq 2$, and, if sufficient analyticity is assumed this can be extended to the interior of the parabola

$$t=(2\mu-i\lambda)^2 \quad \lambda \text{ real}$$

by using the Legendre polynomial expansion of $A_s(s,t)$ instead of the power-series expansion. This should allow one to improve the holomorphy domain previously obtained by Bardakci.[9]

(e) If, in the t channel, poles with angular momentum 0 or 1 occur for $t<4\mu^2$, the conclusions are unchanged because one can merely subtract them. Poles with higher angular momentum will be considered in a separate publication by MacDowell.[10]

Finally, we should mention that we are aware of the fact that the main result of this paper, the number of subtractions for $t<4\mu^2$ is at most two, comes out very naturally in the Regge pole dominance hypothesis, because then, for $0<t<4\mu^2$ the even signature poles should dominate and hence the dominant Regge trajectory cannot cross $l=2$ for $t<4\mu^2$ without producing a pole which was not present by assumption. However, the whole point of our paper is to show that this is true with much more generality.

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[6] See, for instance, J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (American Mathematical Society, New York, 1943), p. 23.

[7] This question will be developed in a forthcoming paper.

[8] M. Froissart, Proceedings of the La Jolla Conference on Weak and Strong Interactions, July 1961 (unpublished).

[9] K. Bardakci, Phys. Rev. **127**, 1832 (1962).

[10] S. W. MacDowell (to be published).